Thm: Let R be a perfectoid Tate algebra.

(i) Let S be a finite étale R-algebra. Then S is a perfectoid Tate algebra and S° is almost finite étale over R°

(ii) Tilting $S \longmapsto S^\flat$ induces an equivalence of categories.

$$R_{fét} := \{\text{finite étale } R\text{-algs}\} \xrightarrow{\sim} \{\text{finite étale } R^\flat\text{-algs}\} =: R^\flat_{fét}.$$

Remark for myself.

How to give S topology:

R_0 ring of definition of R.

π pseudo-uniformizer

pick a finitely generated R_0 module $M \subset S$ such that $M[\frac{1}{\pi}] = S$

Give S the unique linear topology such that M is open with π-adic top.

The weakest topology making R-linear maps $R \to S$ continuous.

We will prove the theorm when $R =$ perfectoid field first and the general theorem will use the field case and similar method.

§ Almost mathmatics.

K perfectoid field.

$m = K^{\circ\circ} = \bigcup_{n \geq 0} \pi^{\frac{1}{p^n}} K$ the subset of topologically nilpotent elements.

Def: An K°-module M is almost zero if and only if $mM = 0$

(Only need to check $\pi^{\frac{1}{p^n}} M = 0$ for all $n \geq 0$.)

Lemma: The full subcategory of almost zero objects in K°-mod is thick.

We can form the localization of category

$$K^\circ\text{-mod} \longrightarrow K^{\circ a}\text{-mod} = K^\circ\text{-mod}/(m\text{-torsion})$$
$$M \longmapsto M^a.$$

K^{oa}-mod has the same object as K°-mod. but we change the hom-sets so that every thing in (m-torsion) is isomorphic to 0.

$$K^\circ\text{-mod} \longrightarrow K^{\circ a}\text{-mod} := K^\circ\text{-mod}/(m\text{-torsion}) \longrightarrow K\text{-mod}$$

"Integral structure" "slightly generic fiber" "generic fiber"

X $U \xhookrightarrow{j} X$ $\{\eta\}$

$j^*: Sh(X) \longrightarrow Sh(U)$ has left and right adjoints $j_!$ and j_*.

The localization functor $(-)^a$ has left and right adjoints $N \longmapsto N_!$ and $N \longmapsto N_*$.

And.
$$(M^a)_* = \mathrm{Hom}_{K^\circ}(m, M)$$
$$(M^a)_! = m \otimes M. \quad \text{(Remark! also exact)}$$

Call M_* the module of <u>almost elements</u> of M.

Lemma. Let M, N be two K°-modules. Then.
$$\mathrm{Hom}_{K^{oa}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(m \otimes M, N)$$

In particularly, $\mathrm{Hom}_{K^{oa}}(X, Y)$ has a natural structure of K^{o} module for any two K^{oa} - modules X and Y.

$$\mathrm{al}\,\mathrm{Hom}(X, Y) :== \mathrm{Hom}_{K^{oa}}(X, Y)^{a}.$$

Def. Let A be any K^{oa}-algebra, M an A-module.

(1) M is flat if $M \otimes_A -$ is exact.

(2) M is almost projective if $\mathrm{al}\,\mathrm{Hom}_A(M, -)$ is exact.

(3) Let $M = N^{a}$ and $A = R^{a}$.

M is almost finitely generated / almost finitely presented / almost free of rank d if for any $\varepsilon \in m$, $\exists$ f.g. / f.P. / free of rank d R-module N_ε and $f_\varepsilon : N_\varepsilon \to N$ with $\ker f$ and $\operatorname{coker} f$ killed by ε.

Def. $f: A \to B$ a map of K^{oa}-algebras. We say f is unramified if there exist an almost element $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu(e) = 1$ and $\ker(\mu)_* \cdot e = 0$.

étale = flat + unramified

finite étale = étale + almost finitely presented.

Lemma. flat + almost f.P. = almost projective + almost f.g.

Remark. $A \to B$ finite étale map of commutative rings.

$\Leftrightarrow$ closed immersion $\mathrm{Spec}\, B \to \mathrm{Spec}\, B \otimes_A B$ is open map.

$\exists$ unique diagonal element $e \in B \otimes_A B$ s.t.

① $e^2 = e$

② $\mu(e) = 1$ for $\mu : B \otimes B \to B$ multiplication

③ $\ker(\mu) \cdot e = 0$.

Now we begin to prove our theorem, first for perfectoid field case.

Lemma. Let M be an A module which is π-adically complete and without [illegible]. Then A-module M is almost free rank of d.

$\iff$ $A/\pi A$-module $M/\pi M$ is almost free rank of d

Proof. $d=1$ to simplify notation.

"$\Leftarrow$" $\exists\, e \in M$ such that kernel and cokernel of

$$\begin{aligned} A/\pi A &\to M/\pi M \\ a &\mapsto ae \end{aligned}$$

killed by $\pi^{\frac{1}{p^n}}$.

Define
$$\begin{aligned} A &\xrightarrow{f} M \\ a &\mapsto ae. \end{aligned}$$

$a \in \ker f \Rightarrow ae = 0 \Rightarrow \pi^{\frac{1}{p^n}} \cdot a \in \pi A \Rightarrow \pi^{\frac{1}{p^n}} a = \pi \cdot b \Rightarrow \pi b e = \pi^{\frac{1}{p^n}} a e = 0$

$\Rightarrow be = 0 \Rightarrow \pi^{\frac{1}{p^n}} b = \pi b_1 \cdots \Rightarrow a = \pi^{(1-\frac{1}{p^n})^k} b_k$ for all k

$\Rightarrow a = 0$

$m \in \operatorname{Coker} f \Rightarrow \pi^{\frac{1}{p^n}} \cdot m = a_1 e + \pi m_1 \Rightarrow \pi^{\frac{1}{p^n}} m_1 = a_2 e + \pi m_2$

$\Rightarrow \pi^{\frac{1}{p^n}} \cdot m = a_1 e + \pi^{(1-\frac{1}{p^n})}(a_2 e) + \pi^{(1-\frac{1}{p^n})} \pi m_2$

$\Rightarrow \pi^{\frac{1}{p^n}} m = 0$ in $\operatorname{Coker} f$.

" $\Rightarrow$ " Similarly.

The same spirit,

Lemma: $A/\pi A \to R/\pi R$ almost finite étale

$\Leftrightarrow$ $A \to R$ almost finite étale.

Prop. Let K be a perfectoid field of char $= p$. L/K a finite field extension. Then $\mathcal{O}_L$ (= integral cloure of $\mathcal{O}_K$ inside L) is almost finite étale. What's more $\mathcal{O}_L$ is almost free rank of d as $\mathcal{O}_K$ module.

Proof.

Recall K perfect. L/K seperable.

$$L \otimes_K L \xrightarrow{\mu} L \xrightarrow{Tr_{L/K}} K$$ is non-degenerate pairing.

$e_1 \cdots e_d \in L$ a K-vector space basis.

$e_1^* \cdots e_d^*$ dual basis then $b = \sum_{i=1}^{d} e_i \, Tr_{L/K}(b e_i^*)$

by theory of seperable extension

$$e := \sum_{i=1}^{d} e_i \otimes e_i^* \in L \otimes_K L$$ is an idempontent.

Frobenius map φ, $x \to x^p$ is automorphism for $K, \mathcal{O}_K, L, \mathcal{O}_L, L\otimes_K L$ and $\varphi(e) = e$

Fix $N >> 0$ such that $\pi^N e_i \in \mathcal{O}_L$ $\quad (e_i \in L = \mathcal{O}_L[\frac{1}{\pi}])$

$$\pi^{2N/p^n} e = \varphi^{-n}(\pi^{2N} e) = \varphi^{-2n}\left(\sum_{j=1}^{d} \pi^N e_j \otimes \pi^N e_j^*\right)$$

$$= \sum_{i=1}^{d} \varphi^{-n}(\pi^N e_i) \otimes \varphi^{-n}(\pi^N e_i^*) \in \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_K.$$

$$e \in (\mathcal{O}_L \otimes \mathcal{O}_K \mathcal{O}_L)_* = \mathrm{Hom}(m, \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L).$$

Hence, $\mathcal{O}_K \to \mathcal{O}_L$ is unramified.

$$f: \mathcal{O}_K^d \longrightarrow \mathcal{O}_L$$

$$(a_1 \cdots a_d) \longrightarrow \sum_{i=1}^{d} \varphi^{-n}(\pi^N e_i) a_i$$

$$g: \mathcal{O}_L \longrightarrow \mathcal{O}_K^d$$

$$b \longmapsto \left(\mathrm{Tr}_{L/K}(b\varphi^{-n}(\pi e_1^*)), \dots\dots \mathrm{Tr}_{L/K}(b\varphi^{-n}(\pi^N e_d^*))\right)$$

$$fg = \pi^{\frac{2N}{p^n}}$$

$$gf = \pi^{\frac{2N}{p^n}}$$

Hence kernel and cokernel of f and g are killed by $\pi^{\frac{2N}{p^n}}$.

Hence $\mathcal{O}_L$ is almost free of rank d.

Similar spirit.

Prop.

R is perfectoid Tate algebra of char $= p$. Let T be a finite étale R algebra. Then T is perfectoid Tate algebra, and T° is almost finite étale over R°

Lemma. Let K be a field which complete under a valuation. $f(x) \in K[x]$ irreducible monic polynomial such that $f(0) \in \mathcal{O}_K$. Then $f(x) \in \mathcal{O}_K[x]$.

Proof: Hensel's lemma.

Lemma. Let K be a perfectoid field. Then

K algebraically closed $\Longleftrightarrow$ K^b is algebraically closed.

Proof: we prove "$\Leftarrow$" direction. ("$\Rightarrow$" direction similarly)

We just need to show for a monic irreducible deg d polynomial $f(x) \in \mathcal{O}_K[x]$. Then $f(x)$ has a root.

Step 1. given $\alpha \in \mathcal{O}_K$, $\exists\, y \in \mathcal{O}_K$ s.t. $|y^d| = |\alpha|$.

Write $\alpha = \pi^m \alpha'$ where $m \geq 0$, $\alpha' \in \mathcal{O}_K \setminus \pi \mathcal{O}_K$.

$\mathcal{O}_{K^b}/\pi^b = \mathcal{O}_K/\pi$ and K^b is algebraically closed.

$\exists\, y \in \mathcal{O}_K$ s.t. $y^d \equiv \alpha' \bmod \pi \mathcal{O}_K$. hence $|y^d| = |\alpha'|$

$\pi^b \in K^b$ has a d^{th} root and $\#: K^b \longrightarrow K$ is multiplicative.

$\left((\pi^b)^{\frac{1}{d}}\right)^\#$ is a d-th root of π.

$y' = \left((\pi^b)^{\frac{m}{d}}\right)^\# \cdot y$ satisfies $|y'|^d = |\alpha|$

Step 2. Given $a \in \mathcal{O}_K$, such that $|f(a)| \leq |\pi|^n$, there exists $\varepsilon \in \mathcal{O}_K$ s.t. $|\varepsilon| \leq |\pi|^{\frac{n}{d}}$ and $|f(a+\varepsilon)| \leq |\pi|^{n+1}$.

$\exists\, y \in \mathcal{O}_K$ such that $|y^d| = |f(a)|$.

$g(X) := y^{-d} f(a + yX)$ is monic and irreducible poly. in $K[X]$

$g(0) = y^{-d} f(a) \in \mathcal{O}_K^* \subset \mathcal{O}_K$.

By lemma again. $g(x) \in \mathcal{O}_K[x]$

$\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$ and $K^\flat$ is algebraically closed,

there $b \in \mathcal{O}_K$ s.t. $g(b) \equiv 0 \mod \pi\mathcal{O}_K$.

Take $\varepsilon = yb$. $|\varepsilon| \le |yb| \le |y| \le |\pi|^{\frac{n}{d}}$

$$|y^{-d} f(a+\varepsilon)| \le \pi.$$

$$|f(a+\varepsilon)| \le |\pi| |y^d| \le |\pi|^{n+1}$$

Thm. Let K be a perfectoid field.

(i) if L/K finite, then L is perfectoid.

(ii)
$$\begin{aligned} \{\text{finite field extension of } K\} &\xrightarrow{\sim} \{\text{finite field extension of } K^\flat\} \\ L &\longmapsto L^\flat \end{aligned}$$

is a degree preserving equivalence of categories

Proof. Recall we already know

$$\begin{aligned} \{\text{perfectoid fields}/K\} &\xrightarrow{\sim} \{\text{perfectoid field}/K^\flat\} \\ L &\longmapsto L^\flat \end{aligned}$$

But we don't know how it preserving finiteness.

Claim 1. If M/K^b is finite extension, then $M^\#/K$ is finite extension of the same deg

Proof. We proved that

$\mathcal{O}_M$ is almost free of rank $[M:K^b]$ as $\mathcal{O}_{K^b}$ module.

$\mathcal{O}_M/\pi_b$ is .... as $\mathcal{O}_{K^b}/\pi^b$ module.

$\cong \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$\mathcal{O}_{M^\#}/\pi$ is .... as $\mathcal{O}_K/\pi$ module.

$\mathcal{O}_{M^\#}$ is .... as $\mathcal{O}_K$ module.

Inverting π tells us $M^\#$ is free K module of rank $[M:K^b]$

Hence we get a fully faithful functor.

$$\{\text{finite extension field}/K^b\} \xrightarrow{\#} \{\text{finite extension field}/K \text{ which is perfectoid}\} \subseteq \{\text{finite extension field}/K\}$$

character p

and above equivalence of categories.

We need to prove the composition is surjective.

Krasner's lemma. (没有足够时间的话只写 Cor)

Let F be a field which is complete wrt. a valuation $|\ |: F \to \mathbb{R}_{\geq 0}$.

let $\alpha, \beta \in F^{sep}$, and let $\alpha_1 = \alpha, \alpha_2 \cdots \alpha_d \in F^{sep}$ be the conjugates of α.

if $|\alpha - \beta| < |\alpha - \alpha_i|$ $i = 2, \cdots d$, then $\alpha \in F(\beta)$

Cor. Let F be a field which is complete wrt a valuation $|\ |: F \to \mathbb{R}_{\geq 0}$ and $F_0 \subseteq F$ a dense subfield. Then F is separably closed $\Leftrightarrow$ F_0 is separably closed.

Let $Q = \left(\left(K^b\right)^{alg}\right)^\wedge$ $\Rightarrow$ Q is perfect and separably closed.

$\Rightarrow$ Q is alg closed perfectoid field.

$\Rightarrow$ $Q^\#$ is alg closed perfectoid field.

Let $Q \supset M \underset{\substack{\text{finite} \\ \text{extension}}}{\supset} K^b$, then $Q^\# \supset M^\# \supset K$.

Take $N := \bigcup_M M^\# \subseteq Q^\#$

$$\mathcal{O}_N/\pi = \varinjlim_M \mathcal{O}_{M^\#}/\pi = \varinjlim_M \mathcal{O}_M/\pi^b = \mathcal{O}_{K^{b\,alg}}/\pi^b = \mathcal{O}_Q/\pi^b = \mathcal{O}_{Q^\#}/\pi$$

Hence N is dense in $Q^\#$

$\Rightarrow$ N is algebraically closed

If L is finite extension of K, then $L \subset N$, $\exists$ M/K^b finite extension such that $L \subset M^\#$. We can take M/K^b to be Galois extension.

Only need to prove ~~the fully faithful functor~~

$$\{\text{subextension of } M/K^b\} \xrightarrow{\#} \{\text{subextension of } M^\#/K\}$$

is surjective.

We know $[M : K^b] = [M^\# : K]$

$\mathrm{Aut}_{K^b}(M) = \mathrm{Aut}_K(M^\#)$ since the functor is fully faithful.

By Galois theory $M^\#/K$ is also Galois extension. They have the same number of subextension.

$\square$

Thm. Let R be a perfectoid Tate algebra.

(i) Let S be a finite étale R-algebra. Then S is a perfectoid Tate algebra and S° is almost finite étale over R°.

(ii)

$$R_{\text{fét}} := \{\text{finite étale } R\text{-alg}\} \xrightarrow{\sim} \{\text{finite étale } R^\flat\text{-alg}\} =: R^\flat_{\text{fét}}$$

Step 1. (Easy for char $=p$. we already mention that)

Lemma. Let T be a finite etale $R^\flat$-algebra. Then T is a perfectoid Tate algebra and T° is almost étale over $R^{\flat\circ}$.

Step 2. Let T be a finite étale $R^\flat$-algebra.

Lemma: $T^\#$ is finite étale over R and $T^{\#\circ}$ is almost finite étale over R°

Proof. By step 1. $R^{\flat\circ} \to T^\circ$ is almost finite étale.

$$\Rightarrow \quad \begin{array}{ccc} R^{\flat\circ}/\pi^\flat & \longrightarrow & T^\circ/\pi^\flat \quad \text{is } \cdots\cdots \\ \| & & \| \\ R^\circ/\pi & \longrightarrow & T^{\#\circ}/\pi \quad \text{is } \cdots\cdots \end{array}$$

$$\Rightarrow \quad R^\circ \longrightarrow T^{\#\circ} \quad \text{is } \cdots\cdots$$

Remark. Similar to field case. we have fully faithful functor.

$$\begin{array}{ccccc} \{\text{finite étale } R^\flat\text{-alg}\} & \xrightarrow{\#} & \left\{\begin{array}{c}\text{perfectoid Tate algebra } S/R \\ \text{such that } S^\circ \text{ is almost} \\ \text{finite étale over } R^\circ\end{array}\right\} & \subseteq & \left\{\begin{array}{c}\text{finite étale} \\ R\text{-alg}\end{array}\right\} \\ \| & & & & \| \\ R^\flat_{\text{fét}} & & & & R_{\text{fét}} \end{array}$$

Step 3. True for field case.

Step 4. gluing local data.

Prop:

① Let A be a ring which is Henselian along an ideal $tA \subseteq A$, where $t \in A$ is a nonzero divisor. Then the base change

$$A[\tfrac{1}{t}]_{\text{fét}} \longrightarrow \hat{A}[\tfrac{1}{t}]_{\text{fét}}$$

is an equivalence of categories, where $\hat{A}$ is the t-adic completion of A

② Let $\varinjlim_i A$ be a filtered colimit of rings. Then

$$\varinjlim_i A_{i\,\text{fét}} \to (\varinjlim_i A_i)_{\text{fét}} \quad \text{is an equivalence of categories}$$

where the left side is a filtered colimit of categories.

Remark: directed system of categories $\{C_i\}_I$.

Functor $F_{ij} : C_i \longrightarrow C_j$ for $i \to j$ in I.

$$2\text{-}\varinjlim C = \begin{cases} \text{objects are objects of any } C_i \\ \mathrm{Hom}(X_i, X_j) = \varinjlim_{i,j\to k} \mathrm{Hom}_{C_k}(F_{ik}(X_i), F_{jk}(X_j)) \quad \text{where } X_i \in C_i,\ X_j \in C_j \end{cases}$$

Let (S, S^+) be any Tate pair and $\pi \in S$ is a pseudo-uniformizer

$x \in X = \mathrm{Spa}(S, S^+)$

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U) \qquad m_{X,x} = \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$$

$$\mathcal{O}^+_{X,x} = \varinjlim_{x \in U} \mathcal{O}^+_X(U) \qquad \{ f \in \mathcal{O}^+_{X,x} \mid |f(x)| < 1 \}$$

$m_{X,x}$ is an idea in $\mathcal{O}^+_{X,x}$.

$$k(x)^+ := \mathcal{O}^+_{X,x} / m_{X,x} \subseteq \mathcal{O}_{X,x} / m_{X,x} =: k(x) = k(x)^+(\tfrac{1}{\pi})$$

Take π-adic completion. Note that $m_{X,x}$ is π-divisible. Hence killed by π-adic completion.

$$\widehat{\mathcal{O}^+_{X,x}} = \left(\mathcal{O}^+_{X,x} / m_{X,x} \right)^\wedge \subseteq \widehat{k(x)}$$

$$\Rightarrow \quad \widehat{k(x)} = \widehat{\mathcal{O}}^+_{X,x}[\tfrac{1}{\pi}]$$

$$\widehat{k(x)}_{f\acute{e}t} = \widehat{\mathcal{O}}^+_{X,x}[\tfrac{1}{\pi}]_{f\acute{e}t} \simeq \mathcal{O}^+_{X,x}[\tfrac{1}{\pi}]_{f\acute{e}t} \simeq \varinjlim_{x \in U} \left(\mathcal{O}_X(U)_{f\acute{e}t} \right)$$

$$\Big\| \qquad \not\simeq$$

$$\left(\varinjlim_{x \in U} \mathcal{O}_X(U) \right)_{f\acute{e}t}$$

Any finite étale $\widehat{k(x)}$-algebra spreads out to a finite étale $\mathcal{O}_X(U)$-algebra for a small rational neighborhood U of x. It is unique in the sense, given another choice, two agree on a smaller neighbourhood.

Back to proof of main theorem.

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & x^b \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ X := \mathrm{Spa}(R, R^\circ) \arrow[r, "\sim"] & X^b := \mathrm{Spa}(R^b, R^{b,\circ}) \\ V \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] \arrow[r, mapsto] & U \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] \end{tikzcd}$$

$$\mathcal{O}_X(V)^b \cong \mathcal{O}_{X^b}(U)$$

$$\begin{tikzcd} \widehat{k(x^b)}_{f\acute{e}t} & \varinjlim_{x^b \in U} \left(\mathcal{O}_{X^b}(U)_{f\acute{e}t}\right) \arrow[l, "\sim"'] \\ \widehat{k(x)}^b_{f\acute{e}t} \arrow[u, equal] \arrow[d, "\#", "\simeq"'] & \varinjlim_{x \in U} \left(\mathcal{O}_X(U)^b_{f\acute{e}t}\right) \arrow[l, "\sim"'] \arrow[u, equal] \arrow[d, "\#"] \\ \widehat{k(x)}_{f\acute{e}t} & \varinjlim_{x \in U} \left(\mathcal{O}_X(U)_{f\acute{e}t}\right) \arrow[l, "\simeq"'] \end{tikzcd}$$

Given a finite étale R-algebra S.

the finite étale $\widehat{k(x)}$ algebra $S \otimes_R \widehat{k(x)}$

can be written as $T_x^{\#} \otimes_{\mathcal{O}_X(U)} \widehat{k(x)}$

for some $\mathcal{O}_X(U_x)^b$-algebra T_x, where U_x is a small neighborhood of x.

Since $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$

$H^i(X, \mathcal{O}_X^+)$ is almost zero

One can glue T_x as we vary x to show

$\#: R^b_{\text{étal}} \longrightarrow R_{\text{étale}}$. hits S, as desired.